**Euler formula:** Euler's formula, named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function.

$e^{i\theta} = \cos\theta + i\sin\theta$

**Multiplication of two complex numbers:**

Let, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

**Then**

$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow z_1 z_2 = r_1 r_2 e^{i\theta_1}.e^{i\theta_2}$

$\Rightarrow z_1 z_2 = r_2 e^{i\theta_1 + i\theta_2}$

$\Rightarrow z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

$\Rightarrow z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Similarly for more than two complex numbers multiplication exists.

**Division of two complex numbers:**

Let, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

**Then**

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}$

$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{e^{i\theta_1}}{e^{i\theta_2}}$

$\Rightarrow z_1 z_2 = \frac{r_1}{r_2} e^{i\theta_1 - i\theta_2}$

$\Rightarrow z_1 z_2 = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Similarly for more than two complex numbers division exists.

**De Moivre's theorem:**

De Moivre's theorem states that, for all real values of n, $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

So for any complex number $z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

**Example**: Evaluate $\left(\sqrt{3} + i\right)^4$ using De Moivre's theorem.

Let, $x + iy = \sqrt{3} + i$ , then

$r = \sqrt{x^2 + y^2}$

$= \sqrt{\left(\sqrt{3}\right)^2 + 1} = \sqrt{4} = 2$

and $\theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

Now

$\left(\sqrt{3} + i\right)^2$

$= [2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})]^4$

$= 2^4\left(\cos 4\frac{\pi}{6} + i\sin 4\frac{\pi}{6}\right)$

$= 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 16\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 8(-1 + i\sqrt{3})$

| **Exercise**<br>Find in the form $z = x + iy$ of the following by making use of De Moivre's theorem. | |
|---|---|
| i) $\left(1 + i\sqrt{3}\right)^5$<br>ii) $(\sqrt{3} - i)^{10}$<br>iii) $\left(1 + i\sqrt{3}\right)^3$<br>i) $(\sqrt{6} - i\sqrt{2})^4$ | ii) $(1 - i)^7$<br>iii) $\left(\sqrt{3} - i\right)^6$<br>iv) $(1 + i)^5$<br>v) $(\sqrt{2} - i)^4$ |

| **** Show that the relation $\left\lvert\frac{z-3}{z+3}\right\rvert = 2$ represents a circle. | |
|---|---|
| Let, $z = x + iy$ then, $\lvert z\rvert = \sqrt{x^2 + y^2}$<br><br>Now $\left\lvert\frac{z-3}{z+3}\right\rvert = 2$<br>$\Rightarrow \left\lvert\frac{x + iy - 3}{x + iy + 3}\right\rvert = 2$<br>$\Rightarrow \left\lvert\frac{(x - 3) + iy}{(x + 3) + iy}\right\rvert = 2$<br>$\Rightarrow \frac{\sqrt{(x - 3)^2 + y^2}}{\sqrt{(x + 3)^2 + y^2}} = 2$ | $\Rightarrow \frac{(x-3)^2+y^2}{(x+3)^2+y^2} = 2^2$<br>$\Rightarrow (x - 3)^2 + y^2 = 4[(x + 3)^2 + y^2]$<br>$\Rightarrow x^2 - 6x + 9 + y^2 = 4[x^2 + 6x + 9 + y^2]$<br>$\Rightarrow x^2 - 6x + 9 + y^2 = 4x^2 + 24x + 36 + 4y^2$<br>$\Rightarrow 4x^2 + 24x + 36 + 4y^2 - (x^2 - 6x + 9 + y^2) = 0$<br>$\Rightarrow 3x^2 + 30x + 27 + 3y^2 = 0$<br>$\Rightarrow 3(x^2 + y^2) + 30x + 27 = 0$<br>$\Rightarrow (x^2 + y^2) + 10x + 9 = 0$<br>$\Rightarrow (x^2 + 10x + 25) + 9 + y^2 = 25$<br>$\Rightarrow (x + 5)^2 + y^2 = 16$ , which is general equation of circle . |

Find the square root of the complex number $(3 + 4i)$.

**Solution:**

$\sqrt{(3 + 4i)}$

$= \pm\sqrt{4 + 2.2i - 1}$

$= \pm\sqrt{4 + 2.2i + i^2}$

$= \pm\sqrt{(2 + i)^2}$

$= \pm(2 + i)$

Find the square root of the complex number $\frac{-1+5i}{2+3i}$

**Solution:**

$$\frac{-1+5i}{2+3i} = \frac{-1+5i}{2+3i} \times \frac{2-3i}{2-3i} = \frac{(-1+5i)(2-3i)}{2^2-(3i)^2}$$

$$= \frac{-2+3i+10i-15i^2}{4-9i^2} = \frac{13i-2+15}{4+9} = \frac{13i+13}{13} \Rightarrow \frac{-1+5i}{2+3i} = 1+i$$

Let, $x+iy = \sqrt{1+i}$, Then

$(x+iy)^2 = 1+i \Rightarrow x^2+2ixy+i^2y^2 = 1+i \Rightarrow x^2-y^2+2ixy = 1+i$

Now equating real and imaginary parts we get,

$$x^2-y^2 = 1 \ldots \ldots . (i)$$

$$2xy = 1 \ldots \ldots \ldots . . (ii)$$

Also

$$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$\Rightarrow (x^2+y^2)^2 = 1^2 + (2xy)^2$$

$$\Rightarrow (x^2+y^2)^2 = 1+1^2$$

$$\Rightarrow (x^2+y^2)^2 = 2$$

$$\Rightarrow x^2+y^2 = \sqrt{2} \ldots \ldots \ldots (iii)$$

**Now**

$$(\mathbf{i}) + (\mathbf{iii}) \Rightarrow 2x^2 = \sqrt{2}+1 \Rightarrow x^2 = \frac{1}{2}\left(\sqrt{2}+1\right) \Rightarrow x = \pm\sqrt{\frac{1}{2}(\sqrt{2}+1)}$$

**And**

$$(\mathbf{iii}) - (\mathbf{i}) \Rightarrow 2y^2 = \sqrt{2}-1 \Rightarrow y^2 = \frac{1}{2}\left(\sqrt{2}-1\right) \Rightarrow y = \pm\sqrt{\frac{1}{2}(\sqrt{2}-1)}$$

**So**

$$x+iy = \pm\left[\sqrt{\frac{1}{2}\left(\sqrt{2}+1\right)} + i\sqrt{\frac{1}{2}\left(\sqrt{2}-1\right)}\right]$$

**Exercise:**

(i) Show that the relation $|z-(-2+i)| = 4$ represents a circle.

(ii) Show that the relation $\left|\frac{Z-1}{z+1}\right| = 6$ represents a circle.

(iii) Show that the relation $\left|\frac{Z-4}{z+4}\right| = 3$ represents a circle.

(iv) Find the square root of the complex number $1+i$

(v) Find the square root of the complex number $8-6i$